Lemma 10.2.17:

Let $\mathcal{G}$ be a finite semi-Cartan graph, X is a point of $\mathcal{G}$. $D \subseteq \Delta^{X re}$

Assume that $\gamma = 0$ or $\gamma \notin \sum_{i \in I} \mathbb{N}_0 \alpha_i$ for any $D' \subseteq \Delta^{X re}$ and $\gamma = \sum_{\beta \in D'} \beta - \sum_{\beta \in D} \beta$

Then $\alpha \in \underline{\underline{D}}$ and $-\alpha \notin D$, $\forall \alpha \in \Delta_+^{X re}$.

$\alpha \in \Delta_+^{X re}$

Proof: Let $D' = D \cup \{\alpha\}$

If $\alpha \notin D$, $\gamma = \sum_{\beta \in D'} \beta - \sum_{\beta \in D} \beta$

$= \alpha \in \sum \mathbb{N}_0 \alpha_i$ ↯

$\Rightarrow \alpha \in D$

Let $D' = D \setminus \{-\alpha\}$

If $-\alpha \in D$ then $\gamma = \cdots$

$= - -\alpha = \alpha$

↯

$\Rightarrow -\alpha \notin D$

Thm 10.2.18

Let $\mathcal{G}$ be a finite semi-Cartan graph, Then $\exists$ a point of $\mathcal{G}$ with a Cartan matrix of finite type.

Proof: Let X be a point of $\mathcal{G}$.

$\mathcal{Y}$ ---.

Let $Y \in \mathcal{Y}$, $D^Y \subseteq \Delta^{Y re}$, $\delta^Y = \sum_{\beta \in D^Y} \beta$

Assume that $\delta^Y \in \sum_{i \in I} \mathbb{N}_0 \alpha_i$ and that $\forall Z \in \mathcal{Y}$, $D' \subseteq \Delta^{Z re}$ $\gamma = \sum_{\beta \in D'} \beta$ implies that $\gamma = \delta^Y$ or $\gamma - \delta^Y \notin \sum_{i \in I} \mathbb{N}_0 \alpha_i$

Since $\mathcal{G}$ is finite,

the set $\boxed{\left| \bigcup_{Y \in \mathcal{Y}} \Delta^{Y re} \right| < \infty}$ Lem 9.1.15

Therefore Y and D^Y exist,

$\underline{\qquad\qquad}\ \delta^{Y}$

Let $x = (x_j)_{j \in I}$, s.t $\delta^{Y} = \sum_{j \in} x_j \alpha_j$

then $x \geq 0$

We show that $A^{Y} x > 0$

Let $i \in I$.

By def $s_i^{Y}(\delta^{Y} - \alpha_i) = \boxed{\alpha_i + \delta^{Y} - \sum_{j \in I} a_{ij}^{Y} x_j \alpha_i} - \delta^{Y}$ $\notin \sum \mathbb{N}_0 X$

Since $s_i^{Y}(\delta^{Y} - \alpha_i)$ is a sum of roots of $r_i(Y) \in Y$

the choice of Y and D^{Y} implies

$$\underline{\sum_{j \in I} a_{ij}^{Y} x_j \geq 1}$$

Thus $\boxed{A_i^{Y}} x > 0$

Now let B be an indecomposable component of A^{Y}, x' then $\boxed{\boxed{B} x' > 0,\ x' \geq 0}$

Hence B is not of affine and indefinite

$\underline{\text{Thm } 10.2.7}$

Hence B is of finite type.

Thus A^{Y} is a Cartan matrix of finite type

by prop 10.2.16

Vinberg: ① indecomposable

② $a_{ij} \leq 0$ $i \neq j$

③ $a_{ij} = 0 \Rightarrow a_{ji} = 0$

Thm 10.2.7 續 Vinberg

finite, affine, indefinite

A finite : symmetrizable

$(\cdot|\cdot)$ p-d

Prop 10.2.16 : A finite

$\Leftrightarrow$ 各个不可分解的

B 皆 finite type

□

10.3 Classification of finite Cartan graph of rank 2

For all integers $1 < i \le n$,

let $V_i : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$

$$V_i(c_1, \dots, c_n) = (c_1, \dots, c_{i-2}, c_{i-1}+1, 1, c_i+1, c_{i+1}, \dots)$$

$i-1 \,,\underset{1}{\wedge}\, i$ $\quad$ c_i 和 c_{i-1} 都要 $+1$

Def 10.3.1

Let A^+ be the smallest subset of $\bigcup_{n\ge 2} \mathbb{N}_0^n$ s.t.

(1) $\underline{(0,0)} \in A^+$

(2) if $(c_1, \dots, c_n) \in A^+$, $1 < i \le n$,

then $V_i(c_1, \dots, c_n) \in A^+$.

For all $n \ge 2$, let $A^+(n)$ denote the set of all $(c_1, \dots, c_n)$

Lem 10.3.2

Let $n \geq 2$, then $\sum_{i=1}^{n} c_i = 3n-6$

for all $(c_1, \dots, c_n) \in A^+(n)$.

Proof: $3n-6$

Example 10.3.3

$A^+(2) = (0,0)$

$A^+(3) = (1,1,1)$

$A^+(4) = \{(2,1,2,1), (1,2,1,2)\}$

$A^+(5) = \{ \dots \}$

Lemma 10.3.4

Let $c \in A^+(n)$, $n \geq 3$

(1) For all $1 \leq i \leq n$, $c_i > 0$ $(1, n-1)$

(2) $\exists\ 1 \leq i < j \leq n$, with $(i,j) \neq (1,n)$

and $c_i = c_j = 1$

(3) If $c_i = c_{i+1} = 1$ for some $1 \leq i < n$

then $n = 3$, $c = (1,1,1)$

Proof: By induction.

We relate sequences in $A^+(n)$ to

triangulations of labeled n-gons.

convex

Def 10.3.5

Let $n \geq 2$, G — n-gons

Enumerate the vertices of G from

1 to n s.t ----.

We write $\mathcal{T}_n$ for the set of triangulations of G with ---

$\forall \underline{T} \in \mathcal{T}_n$ of G, $\forall i \in \{1, \dots, n\}$

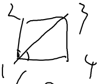
let $\underline{c_i(T)}$ be the number of triangles meeting at i-th vertex

Example 10.3.6

(1) $n=2$ $\quad c_1(T) = c_2(T) = 0$

(2) $n=3$. $\quad c_1(T) = c_2(T) = c_3(T) = 1$

(3) For $n=4$

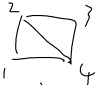


$c_1(T) = 2$

$c_2(T) = 1$

$c_3(T) = 2$

$c_4(T) = 1$



$c_1(T) = 1$

$c_2(T) = 2$

$c_3(T) = 1$

$c_4(T) = 2$

Prop 10.3.7

Let $n \geq 2$, let G — n-gons

Enumerate ---.

Then the map $\mathcal{T}_n \to A^+(n)$

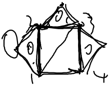
$$T \mapsto (c_1(T), \dots, c_n(T))$$

is a bijection.

Proof: By induction.

$n=3,4$

$$\begin{tikzcd}
\hat{T}_{n-1} \arrow[r, leftrightarrow] \arrow[d] & A^+(n-1) \arrow[d, "V_i\text{ 作用}"] \\
\tilde{T}_n \arrow[r, leftrightarrow] & \boxed{A^+(n)}
\end{tikzcd}$$



Cor 6.3.8 Let $n \geq 2$

(1) Let $\boxed{(c_1, \dots, c_n)} \in A^+(n)$

Then $A^+(n)$ also contains

$\boxed{(c_n, c_{n-1}, \dots, c_1)}$ and

$\boxed{(c_2, c_3, \dots, c_n, c_1)}$

Thus the dihedral group $D_n \subseteq S_n$

acts on $A^+(n)$ by:

$w(c_1, \dots, c_n) = (c_{w^{-1}(1)}, \dots, c_{w^{-1}(n)})$

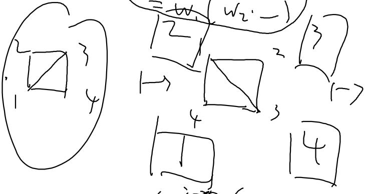
$w \in D_n$

$(w_1 w_2) \cdot \_$

$= w_1(w_2 \cdot \_)$

w

$(1234) \cdot$



C_1 至 C_4

C_2 至 C_1

$(1234)\ (2,1,2,1) = (1,2,1,2)$

$c_1\ c_2\ c_3\ c_4$

(2): ~~

~~Lem~~

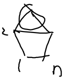
Cor 10.3.9

Let $n \geq 3$, $C = (c_1, \dots, c_n) \in A^+(n)$

Let $1 < i \leq n$, If $c_i = 1$, then $C' \in A^+(n-1)$

$C' = (c_1, \dots, c_{i-2}-1, c_{i+1}-1, c_{i+2}, \dots, c_n)$

Proof: 由三角形剖分



Example 10.3.10

A short calculation shows that

— — — .

Prop 10.3.11

Let $n \geq 3$, Then any sequence $(c_1, \dots, c_n) \in A^+$ contains a subsequence $(c_k)_{i \leq k \leq j}$ with $1 \leq i < j \leq n$ of the form

$(1,1)$, $(1,2,a)$, $(2,1,b)$, $(1,3,1,b)$ or their reversal, $1 \leq a \leq 3$, $3 \leq b \leq 5$

Proof:

Let $C = (c_1, \dots, c_n) \in A^+$, Assume --- not hold.

Then $n \geq 6$ by example 10.3.7.

Since $\underline{n \geq 6}$, Cor (0.3.9) implies that c has no subsequence $\underline{(2,\cancel{1},2)}$.

$\underline{\underline{c(1,1)}}$

Let $\varepsilon_{11} = (1)$ $\varepsilon_{12} = (2,1)$

$\varepsilon_{22} = (1,3,1)$, $\varepsilon_{21} = (1,2)$

$\boxed{E} = \{\varepsilon_{ij} \mid 1 \leq i,j \leq 2\}$

By assumption, both the left and right neighbor of any ε_{ij} with $(i,j) \neq (1,1)$ is at least $\boxed{4}$.

$\varepsilon_{21} : (1,2)$

$(3,(1,2)$ X $b=3$

$(1,2,3)$ X $a=3$

$(2,1,2)$ X

$(1,2,2)$ X $a=2$

$(1,2,1)$ X $a=1$

$(\boxed{1,1},2)$ X

Thus there is a unique decomposition $\underline{d = (d_1, \cdots, d_k)}$ d_i : $\boxed{(a)}$ or ε $\varepsilon \in E$.

$a \geq 2$

s.t (a) $\underline{(\overset{\boxed{\varepsilon_{21}}}{\varepsilon_{11}}, 2), (2, \overset{\boxed{\varepsilon_{12}}}{\varepsilon_{11}}), (\varepsilon_{11}, 3, \overset{\boxed{\varepsilon_{22}}}{\varepsilon_{11}})}$ are not subsequence of d.

for example

$c=(1,4,6,1,3,1,7,2)$

$d=(\varepsilon_{11},4,6,\varepsilon_{22},7,2)$

Since c doesn't satisfy the claim of the prop

we obtain the following information for d:

(b) No two consecutive entries of d $\in E$.

(c) No entry $\varepsilon\in E$ of d is proceed or followed by 2.

then $\varepsilon_{22}=(1,3,1)$ $((2,1,3,1))$ X

$(1,3,1,2)$ X

(d) If (ε_{21},a) or (a,ε_{12})

then $a\geq 4$

(e) If (ε_{i2},b) or (b,ε_{2i}) $i\in\{1,2\}$

then $b\geq 6$

S_{12}: $(2,1,\begin{smallmatrix}1\\2\end{smallmatrix})$ X

$(2,1,\begin{smallmatrix}3\\4\\5\end{smallmatrix})$ X $(2,1,b)$

b6

By iterated application of Cor 10.3.9

迭代

We obtain further reductions of d:

$(\dots, \underline{d_{m-1}}, \Sigma_{ij}, \underline{d_{m+1}}, \dots) \mapsto (\dots, \underline{d_{m-1}-i}, \underline{d_{m+1}-j}, \dots)$

$(\underline{\Sigma_{i2}}, d_2, \dots) \mapsto (\Sigma_{i1}, d_2 - 1, \dots)$

$(\dots, \tilde{d}, \Sigma_{2i}) \mapsto (\dots, \tilde{d} - 1, \Sigma_{1i})$

Let's perform those reductions at all places $1 \le m \le k$ in d, where Σ_{ij} with $1 \le i, j \le 2$ appears.

By (b) and (c),

this result in a unique sequence d' without entries in Σ except maybe at the first or the last.

$$\boxed{\text{Any intermediate entry of } d' \ge 2}$$

;

① any entry $d_m \notin \Sigma$ of d is decreased by at most $\underline{\underline{4}}$

$\underline{\Sigma_{22}}, \underset{-2}{\underset{-2}{d_m}}, \underline{\Sigma_{22}}$

Here if $d_m \ge 6$, ---- ≥ 2

② if $4 \le d_m < 6$

then $d_{m-1} \ne \Sigma_{i2}$ and $d_{m+1} \ne \Sigma_{2i}$ by (e)

Here d_m decreases at most by 2,

≥ 2

③ $d_m = 3$, (a), (d), (e) $\Rightarrow$

at most one of d_{m-1} and d_{m+1} is ≥ 2 and must be ε_{11}.

$\rightsquigarrow$ dense by 1 ≥ 2

④ $d_m = 2$, ≥ 2

Then $\underline{\exists\ d'' =}\ (d_1'', \cdots, d_l'')$ $\to A^+$

$\underline{d_m'' \geq 2,\ 1 < m < l}$

$\underline{d_1'', d_l'' \geq 1}$ 乞 口

A^+的刻~~画~~划

Let $\eta: \mathbb{Z} \longrightarrow SL_2(\mathbb{Z})$

$a \mapsto \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$

(α_1, α_2) is the standard basis of $\mathbb{Z}^2$.

Lemma 5.3.12

Let $n \in \mathbb{N}$, $(c_k)_{1 \leq k \leq n} \in \mathbb{Z}^n$,

for all $1 \leq k \leq n+1$, let $\beta_k = \eta(c_1) \cdots \eta(c_{k-1})(\alpha_1)$

(1) Let $\beta_0 = -\alpha_2$,

Then $\beta_{k+1} = c_k \beta_k - \beta_{k-1}$, $\forall 1 \leq k \leq n$

Proof: $k = 1$: $\beta_1 = \alpha_1$

$\beta_2 = \eta(c_1)(\alpha_1) = \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= c_1\alpha_1 + \alpha_2$$

$$k \geq 2 : \beta_{k+1} = \eta(c_1) \cdots \eta(c_k)(\alpha_1)$$

$$= \eta(c_1) \cdots \eta(c_{k-1})(c_k\alpha_1 + \alpha_2)$$

$$\left(\eta(c_{k-1})(\alpha_2) = \begin{pmatrix} c_{k-1} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\alpha_1 \right)$$

$$= c_k\beta_k - \beta_{k-1}.$$

② If $n \geq 3$ and $c_k = 0$ $\underline{\text{for } 1 \leq k < n}$, then $\beta_\ell \notin \mathbb{N}_0^2$ for some $\underline{1 \leq \ell \leq n}$.

Proof: if $c_1 = 0$, then $\beta_3 = \eta(c_1)(c_2\alpha_1 + \alpha_2$

$$= c_2\alpha_2 - \alpha_1$$

$$\notin \mathbb{N}_0^2$$

if $1 < k < n$, $c_k = 0$,

then by (1) $\beta_{k+1} = -\beta_{k-1}$ $\square$

③ If $c_1 \geq 1$, $c_k \geq 2$, $\forall 1 < k < n$.

then $\beta_k \in \mathbb{N}_0^2$ for all $1 \leq k \leq n$,

and $\underline{\beta_k - \beta_{k-1} \in \mathbb{N}_0^2 \setminus \{0\}}$ $\forall 1 < k \leq n$

Further if $n \geq 2$, then $\beta_{n+1} \in \mathbb{N}_0^2$

or $\beta_{n+1} + \beta_{n-1} \in -\mathbb{N}_0^2$.

Proof: Assume first that $c_k \geq 2$, $\forall 1 \leq k < n$

For all $0 \leq k \leq n$, let $a_k, b_k \in \mathbb{Z}$

s.t. $\beta_k = a_k\alpha_1 + b_k\alpha_2$

$(\beta_0 = \alpha_2)$

We prove by induction on k that

(10.3.3) $\boxed{a_k > b_k \geq 0}$, $\boxed{a_k > a_{k-1}}$

$\boxed{b_k > b_{k-1}}$

$\boxed{a_k - b_k - a_{k-1} + b_{k-1} \geq 0}$ ☆

$\boxed{c_k \geq 2,\ 1 \leq k < n}$

$\underline{c_1 = 1,\quad c_k \geq 2 \quad 1 < k < n}$ 表/m.

(10.3.4) ......

□

if $c_n \geq 1$, $\beta_{n+1} \in \mathbb{N}_0^2$

$$\beta_{n+1} = \underset{\geq 1}{\underline{c_n}}\beta_n - \beta_{n-1} > 0$$

if $c_n \leq 0$ $\quad \underline{\beta_{n+1} + \beta_{n-1} = c_n\beta_n \in -\mathbb{N}_0^2}$

Lemma 10.3.13 □

Let $1 < i \leq n$, $c_1', \ldots, c_n' \in \mathbb{Z}$,

and $(c_1, \ldots, c_{n+1}) = V_i(c_1', \ldots, c_n')$

Thus (1) $\underline{\eta(c_1)\cdots\eta(c_{n+1})}$

$$\underline{=\eta(c_1')\cdots\eta(c_n')}$$

(2) $\beta_k = \beta_k' \quad 1 \le k < i$

$$\beta_k = \beta_{k-1}' \quad i < k \le n+1$$

$$\beta_i = \beta_{i-1}' + \beta_i' = \beta_{i-1} + \beta_{i+1}$$

Proof:

(1): Direct calculation:

$$\underline{\eta(a)\eta(b) = \eta(a+1)\eta(1)\eta(b+1)}$$

17

(2) $\beta_k = \beta_k'$, $1 \le k < i$ obvious

$$\beta_{i+1} = \eta(c_1)\cdots\eta(c_i)(\alpha_1)$$

$$= \eta(c_1')\cdots\eta(c_{i-2}')\eta(c_{i-1}'+1)\eta(1)(\alpha_1)$$

$$= \eta(c_1')\cdots\eta(c_{i-2}')\eta(c_{i-1}'+1)(\alpha_1+\alpha_2)$$

$$= \beta_i' \qquad \eta(c+1)(\alpha_1+\alpha_2) = \eta(c)\alpha_1$$

$\underline{\beta_k = \beta_{k-1}' \text{ for all}}$ $i+1 < k \le n+1$

is obvious

$\underline{(\eta(a)\eta(b) = \eta(a+1)\eta(1)\eta(b))}$

Finally. $\beta_i = \eta(c_1) \cdots \eta(c_{i-1})(\alpha_1)$

$= \cdots = \beta_i' + \beta_{i-1}'$. □

Thm 10.3.14

Let $n \geq 2$, $(c_1, \cdots, c_n) \in \mathbb{Z}^n$.

TFAE: ① $\underline{(c_1, \cdots, c_n) \in \mathcal{A}^+}$

② $\underbrace{\eta(c_1) \cdots \eta(c_n) = -\mathrm{id}}$

$\beta_k = \eta(c_1) \cdots \eta(c_{k-1})(\alpha_1) \in \mathbb{N}_0^2 \quad \forall\, 1 \leq k \leq n$

Proof:

(1) ⇒ (2)

By induction. $n = 2$ ✓: $(c_1, c_2) = (0, 0)$

$\eta^2(0) = -\mathrm{id}$

$\beta_1 = \alpha_1 \quad \beta_2 = \alpha_2 \in \mathbb{N}_0^2$

$n \geq 3$: $\exists\, \boxed{(c_1', \cdots, c_{n-1}')} \in \mathcal{A}^+(n-1)$

s.t. $\forall_i\, (c_1' \cdots c_{n-1}') = (c_1, \cdots, c_n)$

Then (2) follows from 10.3.13

(2) ⇒ (1) :

If $n=2$, then $-\mathrm{id} = \eta(c_1)\eta(c_2)$

$$= \begin{pmatrix} c_1c_2-1 & -c_1 \\ c_2 & -1 \end{pmatrix}$$

$\Rightarrow c_1 = c_2 = 0$, so $(c_1, c_2) \in A^{+}$.

Assume $n \geq 3$:

$\forall 1 \leq k < n$, $\boxed{c_k \beta_k = \beta_{k-1} + \beta_{k+1}}$

$(\beta_0 = -\alpha_2)$

Since $\beta_\ell \in \mathbb{N}_0^2$ $\forall 1 \leq \ell \leq n$,

we obtain $\underline{c_k > 0, \forall 1 < k < n}$

$\underline{c_1 \geq 0}$ (β_2是正根)

Further $c_1 > 0$, by lem 10.3.12(2)

Since $\eta(c_1) \cdots \eta(c_n) = -\mathrm{id}$

we have $\beta_{n+1} = -\alpha_1$

$\Rightarrow \boxed{\exists\, i \in \{2, \cdots, n-1\} \quad c_i = 1}$ by $\underline{10.3.12(3)}$

Let $(c_1', \dots, c_{n-1}') \in \mathbb{Z}^{n-1}$

s.t $\forall i$ $(c_1', \dots, c_{n-1}') = (c_1, \dots, c_n)$

Then $\underline{\eta(c_1') \cdots \eta(c_{n-1}') = id}$

Since $\{0,\beta_1\}$

$\beta_k' = \eta(c_1') \cdots \eta(c_{k-1}')(\alpha_1)$

$\forall\, 1 \le k \le n-1 \in \mathbb{N}_0^2$

$\Rightarrow (c_1', \dots, c_{n-1}') \in \mathfrak{A}^+(n-1)$

$\Rightarrow (c_1, \dots, c_n) \in \mathfrak{A}^+$.

$\square$